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Stability and Estimates for the Convergence of Solutions for Systems Involving Quadratic Terms with Constant Deviating Arguments

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Abstract—In this paper, we shall consider systems of differential equations involving quadratic terms with constant deviating arguments. Stability of the systems is investigated by employing the second Lyapunov's method with a special rational function. This function provides convergence rate estimates as well as offers explicit sufficient conditions for the asymptotic stability of solutions.

Keywords—Quadratic terms, Deviating arguments, Asymptotic stability, Lyapunov's second method.

1. INTRODUCTION

A number of mathematical models in medical sciences and ecology have been recently described through differential equations involving quadratic terms [1–4]. This is due to the fact that quadratic terms are more adequate to describe the mutual influence of individuals. Further, an important fact in biological processes is the time required for maturity of individuals, which is best modeled by differential, difference, and difference-differential equations [5–7]. One of the key problems in the investigation of the dynamics of the systems is the identification of the stability of a stationary state that can be reduced to the zero solution. In this paper, we shall consider a system of differential equations involving a quadratic term with a constant deviating argument. A unified form is proposed for representing a system by modular rectangular matrices. It is assumed that the linear approximation is asymptotically stable. This is a natural restriction, because the addition of quadratic terms do not help in improving the stability of

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the zero solution of the system. A guaranteed stability domain for the zero equilibrium state is obtained. In our investigation, a special Lyapunov's rational function is employed. Finally, we remark that asymptotic stability conditions that depend on the deviating arguments have been recently obtained in [8,9].

2. MAIN RESULTS

In this paper, we shall consider the following differential system involving a quadratic term with a constant deviating argument:

$$\dot{x}(t) = A_1x(t) + A_2x(t-\tau) + X^\top(t)B_1x(t) + X^\top(t)B_2x(t-\tau) + X^\top(t-\tau)B_3x(t-\tau). \quad (1)$$

Here A_1, A_2 are square matrices with constant coefficients, $X(t)$, B_j, $j = 1,2,3$ are rectangular $n^2 \times n$ matrices

$$X^\top(t) = \left\{X_1^\top(t), X_2^\top(t), \dots, X_n^\top(t)\right\}, \qquad B_j^\top = \left\{B_1^j, B_2^j, \dots, B_n^j\right\},$$

$X_i^\top(t)$, $i = 1,\dots,n$ is a square matrix whose i^{th} row has the vector $x^\top(t) = \{x_1(t), x_2(t), \dots, x_n(t)\}$, and all other elements are zero. B_i^j, $i = 1,\dots,n$, $j = 1,2,3$ are symmetric matrices that identify the components of the system; see [8,9].

In what follows, we shall assume that the matrix $A = A_1 + A_2$ is asymptotically stable, so that the solution $x(t) \equiv 0$ of the nonlinear system (1) is also asymptotically stable. We shall find the radius R of the ball U_R that lies in the domain of asymptotical stability of the solution $x(t) \equiv 0$. We shall also obtain an estimate for convergence of solutions $x(t)$ with the initial data in this ball. For this, we shall use the second Lyapunov's method with a special type rational function

$$v(x,t) = e^{\gamma t} x^\top H x \left[1 - \xi\sqrt{x^\top H x}\right]^{-2}. \quad (2)$$

The parameters $\gamma > 0$ and $\xi > 0$, which are defined below, furnish more precise estimates for the convergence of solutions $x(t)$ of system (1) than those that can be inferred from a usual quadratic type Lyapunov's function. The symmetric positive definite matrix H is chosen in such a way that it allows for the matrix $C = -A^\top H - HA$ to be positive definite [10].

To clarify our approach, we define a level surface of the Lyapunov's function (2) as ∂v_t^α, and a domain which contains it as v_t^α,

$$\partial v_t^\alpha = \{(x,t) : v(x,t) = \alpha\}, \qquad v_t^\alpha = \{(x,t) : v(x,t) < \alpha\}.$$

As usual, we define $\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ as the extremal eigenvalues of the corresponding symmetric positive definite matrices, and will denote by $\varphi(H) = \sqrt{\lambda_{\max}(H)/\lambda_{\min}(H)}$. We also recall the following estimates for the quadratic form $x^\top Hx$:

$$\lambda_{\min}(H)|x|^2 \le x^\top Hx \le \lambda_{\max}(H)|x|^2. \quad (3)$$

We begin by proving the following lemma.

LEMMA 1. *Let the integral curve $(x(t),t)$ of system (1) be such that $(x(t),t) \in \partial v_t^\alpha$, $(x(s),s) \in v_t^\alpha$, $s < t$, and $1 - \xi\sqrt{x^\top(s)Hx(s)} > 0$, $s < t$. Then, for this curve, the following estimate holds:*

$$|x(s)| < M(t-s)\sqrt{\varphi(H)}\,|x(t)|, \quad (4)$$

where

$$M(t-s) = e^{\gamma(t-s)/2}\left\{1 + \xi\sqrt{x^\top(t)Hx(t)}\left[e^{\gamma(t-s)/2} - 1\right]\right\}^{-1}.$$

PROOF. From the conditions of Lemma 1, we have

$$\frac{e^{\gamma s} x^{\top}(s) H x(s)}{\left[1-\xi\sqrt{x^{\top}(s)Hx(s)}\right]^2} < \frac{e^{\gamma t} x^{\top}(t) H x(t)}{\left[1-\xi\sqrt{x^{\top}(t)Hx(t)}\right]^2}.$$

Thus, it follows that

$$e^{\gamma s/2}\sqrt{x^{\top}(s)Hx(s)}\left[1+\xi\sqrt{x^{\top}(t)Hx(t)}\left(e^{\gamma(t-s)/2}-1\right)\right] < e^{\gamma t/2}\sqrt{x^{\top}(t)Hx(t)}.$$

Therefore, for $s < t$,

$$\sqrt{x^{\top}(s)Hx(s)} < e^{\gamma(t-s)/2}\sqrt{x^{\top}(t)Hx(t)}\left[1+\xi\sqrt{x^{\top}(t)Hx(t)}\left(e^{\gamma(t-s)/2}-1\right)\right]^{-1}.$$

Estimate (4) now follows from the inequalities (3).

NOTE. If $s = t - 2\tau$, then

$$M(2\tau) = e^{\gamma\tau}\left\{1+\xi\sqrt{x^{\top}(t)Hx(t)}\left[e^{\gamma\tau}-1\right]\right\}^{-1}. \tag{5}$$

LEMMA 2. *Let $\alpha > 0$, $\gamma > 0$, $\xi > 0$ be such that the integral curve $(x(s), s) \in v_t^{\alpha}$, $s \geq 0$, and $1-\xi\sqrt{x^{\top}(s)Hx(s)} > 0$. Then,*

$$|x(s)| < \sqrt{\alpha}e^{-\gamma s/2}\left\{\sqrt{\lambda_{\min}}\left[1+\xi\sqrt{\alpha}e^{-\gamma s/2}\right]\right\}^{-1}. \tag{6}$$

PROOF. From the conditions of Lemma 2 and the form of the function $v(x,t)$, we have

$$e^{\gamma s}x^{\top}(s)Hx(s)\left[1-\xi\sqrt{x^{\top}(s)Hx(s)}\right]^{-2} < \alpha.$$

Thus, it follows that

$$e^{\gamma s/2}\sqrt{x^{\top}(s)Hx(s)} < \sqrt{\alpha} - \sqrt{\alpha}\xi\sqrt{x^{\top}(s)Hx(s)}.$$

Now, since $1+\xi\sqrt{\alpha} > 0$, we obtain

$$\sqrt{x^{\top}(s)Hx(s)} < \sqrt{\alpha}\left[e^{\gamma s/2}+\xi\sqrt{\alpha}\right]^{-1}.$$

Next, using inequalities (3), we get

$$\sqrt{\lambda_{\min}(H)}\,|x(s)| < \sqrt{\alpha}\,e^{-\gamma s/2}\left(1+\xi\sqrt{\alpha}\,e^{-\gamma s/2}\right)^{-1}.$$

From this inequality, (6) is immediate.

LEMMA 3. *Let $L^* > 0$ and $\Psi(\gamma)$ be a concave monotone increasing function for $\gamma \geq 0$ and satisfy the conditions $\Psi(0) = 0$, $\lim_{\gamma\to+\infty}\Psi(\gamma) = \Psi^*$. Then, on the interval $0 \leq \gamma \leq \gamma_0$, where*

$$\gamma_0 = L^*\left[\Psi'(0)+\lambda_{\max}(H)\right]^{-1}, \tag{7}$$

the following inequality holds:

$$L^* - \Psi(\gamma) \geq \gamma\lambda_{\max}(H). \tag{8}$$

PROOF. The function $L^* - \Psi(\gamma)$ in the variable γ is convex and monotonically decreasing, and satisfies the conditions

$$L^* - \Psi(0) > 0, \qquad \lim_{\gamma\to+\infty}\left[L^* - \Psi(\gamma)\right] = L^* - \Psi^*.$$

Inequality (8) defines a set of values $\gamma \geq 0$ such that the curve $L^* - \Psi(\gamma)$ is above the line $\gamma\lambda_{\max}(H)$. On this curve $L^* - \Psi(\gamma)$, the equation of the line passing through the point $(0, L^*)$ with angle $-\Psi'(0)$ can be written as

$$\bar{\Psi}(\gamma) = -\Psi'(0)\gamma + L^*.$$

Now for $0 \leq \gamma \leq \gamma_0$, such that

$$-\Psi'(0)\lambda + L^* \geq \gamma\lambda_{\max}(H),$$

inequality (8) holds. Further, the last inequality above leads to (7).

LEMMA 4. *Let $(x(s), s) \in v_t^\alpha$, $(x(t), t) \in \partial v_t^\alpha$, for $-\tau < s < t$, $t > \tau$. Then,*

$$|x(t) - x(t-\tau)| \\ \leq \left[|A_1| + |A_2| + (|B_1| + |B_2| + |B_3|)\sqrt{\varphi(H)}\, M(2\tau)\, |x(t)|\right] \sqrt{\varphi(H)}\, M(2\tau)\, |x(t)|\, \tau, \tag{9}$$

where $M(2\tau)$ is defined in (5).

PROOF. In integral form, system (1) can be written as

$$x(t) = x(t-\tau) + \int_{t-\tau}^{t} \left[A_1 x(s) + A_2 x(s-\tau) + X^\top(s) B_1 x(s) \right. \\ \left. + X^\top(s) B_2 x(s-\tau) + X^\top(s-\tau) B_3 x(s-\tau)\right] ds.$$

Thus, we have

$$|x(t) - x(t-\tau)| \leq \int_{t-\tau}^{t} \left[|A_1|\, |x(s)| + |A_2|\, |x(s-\tau)| + |B_1|\, |x(s)|^2 \right. \\ \left. + |B_2|\, |x(s)|\, |x(s-\tau)| + |B_3|\, |x(s-\tau)|^2\right] ds.$$

Now, since in view of Lemma 1, $|x(s)| < M(t-s)\sqrt{\varphi(H)}\, |x(t)|$, the following holds:

$$|x(t) - x(t-\tau)| \leq \Big\{|A_1|\, M(\tau) + |A_2|\, M(2\tau) \\ + \left[|B_1|\, M^2(\tau) + |B_2|\, M(\tau) M(2\tau) \right. \\ \left. + |B_3|\, M^2(2\tau)\right] \sqrt{\varphi(H)}\, |x(t)|\Big\} \sqrt{\varphi(H)}\, |x(t)|\, \tau.$$

Finally, from the hypothesis $1 - \xi\sqrt{x^\top(t) H x(t)} > 0$. Therefore, $M(\tau) < M(2\tau)$, and from this (9) is immediate.

To obtain stability conditions, it is necessary to provide an estimate for the maximal value of the deviation between the solutions and the equilibrium point. In contrast with the linear case, systems involving square terms possess a nonextensibility property; i.e., their solutions may become infinitely large during a finite time. This is, in fact, also the case for simple scalar equations. We shall find conditions on the parameters of the system, the delay τ, and of the initial perturbation $\|x(0)\|_\tau$ which will allow us to estimate $\|x(t)\|_\tau$ on the interval $0 \leq t \leq \tau$. For this, we introduce the following:

$$\|x(t)\|_\tau = \max_{-\tau \leq s \leq 0} \{|x(t+s)|\}, \qquad L = |A_1| + |B_2|\, \tau, \\ P = \left[1 + (|A_2| + |B_3|\, \|x(0)\|_\tau)\, \tau\right] \|x(0)\|_\tau. \tag{10}$$

LEMMA 5. *Let $\|x(0)\|_\tau$ and $\tau > 0$ be such that*

$$L + P|B_1|\left(1 - e^{L\tau}\right) > 0. \tag{11}$$

Then, for an arbitrary solution $x(t)$ of system (1) for $0 < t \leq \tau$, the inequality $|x(t)| \leq z(t)$ holds, where

$$z(t) = PLe^{Lt}\left[L + P\,|B_1|\left(1 - e^{Lt}\right)\right]^{-1}. \tag{12}$$

PROOF. In integral form, system (1) can be written as

$$x(t) = x(0) + \int_{0}^{t} \left[A_1 x(s) + A_2 x(s-\tau) + X^\top(s) B_1 x(s) \right. \\ \left. + X^\top(s) B_2 x(s-\tau) + X^\top(s-\tau) B_3 x(s-\tau)\right] ds.$$

Thus, for $0 < t \le \tau$, the following holds:

$$|x(t)| \le [1 + (|A_2| + |B_3| \, \|x(0)\|_\tau) \tau] \, \|x(0)\|_\tau + \int_0^t \left[(|A_1| + |B_2| \, \|x(0)\|_\tau) \, |x(s)| + |B_1| \, |x(s)|^2 \right] ds.$$

Now recalling Bihari's Lemma [11], if

$$u(t) \le P + \int_0^t \Phi\left(u(\xi)\right) d\xi, \qquad u(t) \ge 0,$$

then $u(t) \le \Psi^{-1}(t)$, where $\Psi^{-1}(t)$ is the inverse function of

$$\Psi(u) = \int_P^u \frac{d\zeta}{\Phi(\zeta)};$$

in view of (11), we obtain

$$\Phi(\zeta) = \zeta \left(L + |B_1| \, \zeta\right), \qquad \Psi(\zeta) = \frac{1}{L} \ln \left[\frac{u \left(L + P \, |B_1|\right)}{P \left(L + u \, |B_1|\right)} \right].$$

Therefore,

$$\Psi^{-1}(t) = \frac{PLe^{Lt}}{L + P \, |B_1| \, (1 - e^{Lt})}$$

and from this the lemma is immediate.

Thus, any solution $x(t)$ of system (1) does not tend to infinity during the time interval $0 \le t \le \tau$ provided (11) holds.

To state our results in a compact form, we introduce the following:

$$\begin{aligned}
\Psi(\gamma) = \lambda_{\min}(C) \left(1 - \frac{\tau}{\tau_0}\right) - \frac{z(\gamma)\sqrt{\varphi(H)}}{1 - \zeta z(\gamma)} & \Bigg[|HA_2| \Bigg(|A_1| + |A_2| \\
& + \frac{2 + (\zeta + 1) z(\gamma)}{1 + \zeta z(\gamma)} \sum_{i=1}^{3} |B_i| \, \bar{R} \zeta \varphi(H) \Bigg) \tau + |H| \, |B_2| \, \bar{R} \zeta \sqrt{\varphi(H)} \\
& + |H| \, |B_3| \, \bar{R} \zeta \varphi(H) \frac{2 + (\zeta + 1) z(\gamma)}{1 + \zeta z(\gamma)} \Bigg], \qquad z(\gamma) = e^{\gamma\tau} - 1, \\
\gamma_0 = \frac{\lambda_{\min}(C) \left(1 - \tau/\tau_0\right)}{\Psi'(0) + \lambda_{\max}(H)}, & \qquad \tau_0 = \frac{\lambda_{\min}(C) \left(1 - \tau/\tau_0\right)}{2 \, |HA_2| \left(|A_1| + |A_2|\right) \sqrt{\varphi(H)}},
\end{aligned} \tag{13}$$

here $\Psi'(0)$ is a derivative of $\Psi(\gamma)$ at $\gamma = 0$,

$$\begin{aligned}
\bar{R} &= \frac{\lambda_{\min}(C) \left(1 - \tau/\tau_0\right)}{2 \sum_{i=1}^{3} \left[|HA_2| \, |B_i| \, \varphi^{3/2}(H) \tau + \lambda_{\max}(H) \, |B_i| \, \varphi^{i/2}(H) \right]}, \\
\delta\left(\bar{R}, \tau\right) &= \begin{cases} N\left(\bar{R}, \tau\right) \left[1 + |A_2| \, \tau\right]^{-1}, & |B_3| = 0, \\ 2N\left(\bar{R}, \tau\right) \left[\sqrt{\left(1 + |A_2| \, \tau\right)^2 + 4N\left(\bar{R}, \tau\right) |B_3| \, \tau} - \left(1 + |A_2| \, \tau\right) \right]^{-1}, & |B_3| \neq 0, \end{cases} \\
N\left(\bar{R}, \tau\right) &= \frac{\left(|A_1| + |B_2| \, \tau\right) \bar{R} / \sqrt{\varphi(H)}}{\left[|A_1| + |B_2| \, \tau + |B_1| \, \bar{R} / \sqrt{\varphi(H)} \right] \exp\left\{ \left(|A_1| + |B_2| \, \tau\right) \tau \right\} - |B_1| \, \bar{R} / \sqrt{\varphi(H)}}.
\end{aligned} \tag{14}$$

THEOREM 1. *Let $A = A_1 + A_2$ be an asymptotically stable matrix. Then, for $\tau < \tau_0$, the solution $x(t) \equiv 0$ of system (1) is asymptotically stable. The stability domain contains the ball U_R with the radius $R = \delta(\bar{R}, \tau)$. For an arbitrary solution $x(t)$, such that $z(\tau) < \bar{R}\zeta$, $0 < \zeta < 1$ the following convergence estimate holds:*

$$|x(t)| \le \begin{cases} z(t), & 0 \le t \le \tau, \\ \dfrac{\bar{R}\zeta\sqrt{\varphi(H)}\, z(\tau)\exp\{-\gamma t/2\}}{\bar{R}\zeta - z(\tau)\left[1 - \exp\{-\gamma t/2\}\right]}, & t > \tau, \end{cases} \tag{15}$$

where $0 \le \gamma \le \gamma_0$, and $z(t)$ is defined in (12).

PROOF. It follows from (11),(13), and $\delta(\bar{R}, \tau)$, $N(\bar{R}, \tau)$ given in (14), that if the initial condition of the solution $x(t)$ satisfy the inequality $\|x(0)\|_\tau < R$, where $R = \delta(\bar{R}, \tau)$, then

$$\max_{0 \le t \le \tau} \{|x(t)|\} \le z(\tau) \le \frac{\zeta \bar{R}}{\sqrt{\varphi(H)}}, \qquad 0 < \zeta < 1.$$

Consider a level surface ∂v_t^α such that

$$\sqrt{\alpha} = \frac{\sqrt{\lambda_{\max}(H)}\, z(\tau)}{1 - \xi\sqrt{\lambda_{\min}(H)}\, z(\tau)}, \qquad \xi = \frac{1}{\bar{R}\sqrt{\lambda_{\min}}}. \tag{16}$$

From the choice of α, the integral curve $(x(t), t)$ for $0 \le t \le \tau$ lies in the domain of v_t^α, i.e., $(x(t), t) \in v_t^\alpha$. We shall show that this also holds for $t > \tau$. Let us, on the contrary, for some $T > \tau$ have $(x(T), T) \in \partial v_t^\alpha$. Then conditions of Lemmas 1 and 4, and hence, the inequalities (4),(9) hold.

Let us find the total derivative of the Lyapunov's function defined in (2) along system (1). After using the transformations, we have

$$\dot{v}(x(t), t) = e^{\gamma t}\left(1 - \xi\sqrt{x^\top(t) H x(t)}\right)^{-3}$$
$$\left\{\gamma\left(1 - \xi\sqrt{x^\top(t) H x(t)}\right) x^\top(t) H x(t) - \frac{d}{dt} x^\top(t) H x(t)\right\}.$$

Further, along the solutions of system (1),

$$\begin{aligned} \frac{d}{dt} x^\top(t) H x(t) = & -x^\top(t) C x(t) + 2x^\top(t) H \left\{A_2\left[x(t-\tau) - x(t)\right]\right. \\ & + X^\top(t) B_1 x(t) + X^\top(t) B_2 x(t-\tau) \\ & \left. + X^\top(t-\tau) B_3 x(t-\tau)\right\}. \end{aligned}$$

Now using inequalities (3) and expression (5), we obtain the following estimate for the total derivative of the function $v(x(t), t)$ at $t = T$:

$$\begin{aligned} \dot{v}\left(x(T), T\right) \le -e^{\gamma T} & \left(1 - \xi\sqrt{x^\top(T) H x(T)}\right)^{-3} \left\{-\gamma\left(1 - \xi\sqrt{x^\top(T) H x(T)}\right)\right. \\ & \times \lambda_{\max}(H)\, |x(T)|^2 + \lambda_{\min}(C)\, |x(T)|^2 \\ & - 2\,|HA_2| \left[|A_1| + |A_2| + (|B_1| + |B_2| + |B_3|)\, M(2\tau)\sqrt{\varphi(H)}\; |x(T)|\right] \\ & \times M(2\tau)\sqrt{\varphi(H)}\; |x(T)|^2\, \tau \\ & \left. -2|H| \left[|B_1| + |B_2|\, M(2\tau)\sqrt{\varphi(H)} + |B_3|\, M^2(2\tau)\varphi(H)\right] |x(T)|^3 \right\}, \end{aligned}$$

which is better written as

$$\begin{aligned}
\dot{v}\left(x(T),T\right) \le -e^{\gamma T}\left(1-\xi\sqrt{x^{\top}(T)Hx(T)}\right)^{-3}\Bigg\{ & -\gamma\left(1-\xi\sqrt{x^{\top}(T)Hx(T)}\right) \\
& \times \lambda_{\max}(H)+\lambda_{\min}(C)-2\left|HA_2\right|\left(|A_1|+|A_2|\right)\sqrt{\varphi(H)}\,\tau \\
& -2\Big[\,|HA_2|\left(M(2\tau)+1\right)\left(|B_1|+|B_2|+|B_3|\right)\varphi(H)\tau \\
& +|H|\left(|B_1|+|B_2|\right)\sqrt{\varphi(H)}\,+|B_3|\,\varphi(H)\Big]\,|x(T)| \\
& -2\Big[|HA_2|\left(|A_1|+|A_2|\right)+\left(|B_1|+|B_2|+|B_3|\right)\sqrt{\varphi(H)}\,|x(T)|\Big]\sqrt{\varphi(H)}\,\tau \\
& +\Big[|H|\left(|B_2|+|B_3|\left(M(2\tau)t+1\right)\sqrt{\varphi(H)}\right)|x(T)|\Big]\left(M(2\tau)-1\right)\Bigg\}\,|x(T)|^2 .
\end{aligned}$$

Next we assume that the delay τ satisfies the condition $\tau<\tau_0$, where τ_0 is defined in (13). Then using the definition $\bar{R}$ given in (14), we obtain

$$\begin{aligned}
\dot{v}\left(x(T),T\right) \le -e^{\gamma T}\left(1-\xi\sqrt{x^{\top}(T)Hx(T)}\right)^{-3}\Bigg\{ & -\gamma\left(1-\xi\sqrt{x^{\top}(T)Hx(T)}\right) \\
& \times \lambda_{\max}(H)+\lambda_{\min}(C)\left(1-\frac{\tau}{\tau_0}\right)\left(1-\frac{|x(T)|}{\bar{R}}\right) \\
& -2\Big[\,|HA_2|\Big(\,|A_1|+|A_2|+\left(M(2\tau)+1\right)\left(|B_1|+|B_2|\right. \\
& \left.+|B_3|\right)\sqrt{\varphi(H)}\,|x(T)|\Big)\sqrt{\varphi(H)}\,\tau \\
& +|H|\left(|B_2|+|B_3|\left(M(2\tau)+1\right)\sqrt{\varphi(H)}\right)\sqrt{\varphi(H)}\,|x(T)|\Big] \\
& \times\left(M(2\tau)-1\right)\Bigg\}\,|x(T)|^2 ,
\end{aligned}$$

which provides

$$\begin{aligned}
\dot{v}(x(T),T) \le -e^{\gamma T}\left(1-\xi\sqrt{x^{\top}(T)Hx(T)}\right)^{-2}\Bigg\{ & -\gamma\lambda_{\max}(H) \\
& +\frac{\lambda_{\min}(C)\left(1-\tau/\tau_0\right)\left(1-|x(T)|/\bar{R}\right)}{1-\xi\sqrt{x^{\top}(T)Hx(T)}}-2\frac{M(2\tau)-1}{1-\xi\sqrt{x^{\top}(T)Hx(T)}} \\
& \times\Big[|HA_2|\Big(\,|A_1|+|A_2|+\left(M(2\tau)+1\right)\left(|B_1|+|B_2|\right. \\
& \left.+\ |B_3|\right)\sqrt{\varphi(H)}\,|x(T)|\Big)\sqrt{\varphi(H)}\,\tau \\
& +|H|\left(|B_2|+|B_3|\right)\left(M(2\tau)+1\right)\sqrt{\varphi(H)}\,|x(T)|\Big]\Bigg\}\,|x(T)|^2 .
\end{aligned}$$

By using the definition $M(2\tau)$ given in (5), we have

$$\frac{M(2\tau)-1}{1-\xi\sqrt{x^{\top}(T)Hx(T)}}=\frac{e^{\gamma\tau}-1}{1+\xi\sqrt{x^{\top}(T)Hx(T)}\,\left(e^{\gamma\tau}-1\right)}.$$

Therefore, it follows that

$$\dot{v}\left(x(T),T\right) \le -e^{\gamma T}\left(1-\xi\sqrt{x^{\top}(T)Hx(T)}\right)^{-2}\Bigg\{-\gamma\lambda_{\max}(H)$$

$$
\begin{aligned}
&+\frac{\lambda_{\min}(C)\left(1-\tau/\tau_0\right)\left(1-|x(T)|/\bar{R}\right)}{1-\xi\sqrt{x^\top(T)Hx(T)}}\\
&-2\frac{2(e^{\gamma\tau}-1)}{1+\xi\sqrt{x^\top(T)Hx(T)}\,(e^{\gamma\tau}-1)}\\
&\times\Big[|HA_2|\Big(|A_1|+|A_2|+(M(2\tau)+1)\\
&\times(|B_1|+|B_2|+|B_3|)\sqrt{\varphi(H)}\,|x(T)|\Big)\sqrt{\varphi(H)}\,\tau\\
&+|H|\left(|B_2|+|B_3|\right)(M(2\tau)+1)\sqrt{\varphi(H)}\,|x(T)|\Big]\Bigg\}|x(T)|^2.
\end{aligned}
$$

Let $\xi = 1/\bar{R}\sqrt{\lambda_{\min}(H)}$, so that

$$
1-\xi\sqrt{x^\top(T)Hx(T)} = \frac{\bar{R}\sqrt{\lambda_{\min}(H)}-\sqrt{x^\top(T)Hx(T)}}{\bar{R}\sqrt{\lambda_{\min}(H)}}.
$$

We also note that for $|x(T)| < \bar{R}/\sqrt{\varphi(H)}$, the following holds:

$$
1-\xi\sqrt{x^\top(T)Hx(T)} > 0.
$$

Now since

(1)

$$
\frac{1-|x(T)|/\bar{R}}{1-\xi\sqrt{x^\top(T)Hx(T)}} = \frac{\left(\bar{R}-|x(T)|\right)\sqrt{\lambda_{\min}(H)}}{\bar{R}\sqrt{\lambda_{\min}(H)}-\sqrt{x^\top(T)Hx(T)}} \geq 1,
$$

(2)

$$
\begin{aligned}
\left(1-\xi\sqrt{x^\top(T)Hx(T)}\right)^{-1} &= \left(1-\frac{\sqrt{x^\top(T)Hx(T)}}{\bar{R}\sqrt{\lambda_{\min}(H)}}\right)^{-1} \geq \frac{\bar{R}}{\bar{R}-|x(T)|}\\
&= \left(1-\frac{|x(T)|}{\bar{R}}\right)^{-1},
\end{aligned}
$$

(3)

$$
\begin{aligned}
\left[1+\xi\sqrt{x^\top(T)Hx(T)}\,(e^{\gamma\tau}-1)\right]^{-1} &= \left[1+\frac{\sqrt{x^\top(T)Hx(T)}}{\bar{R}\sqrt{\lambda_{\min}(H)}}\,(e^{\gamma\tau}-1)\right]^{-1}\\
&\leq \left[1+\frac{(e^{\gamma\tau}-1)\,|x(T)|}{\bar{R}}\right]^{-1},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\dot{v}\left(x(T),T\right) \leq{}& -e^{\gamma T}\left(1-\frac{|x(T)|}{\bar{R}}\right)^{-2}\Bigg\{-\gamma\lambda_{\max}(H)+\lambda_{\min}(C)\left(1-\frac{\tau}{\tau_0}\right)\\
&-\frac{2\left(e^{\gamma\tau}-1\right)\sqrt{\varphi(H)}}{1+(e^{\gamma\tau}-1)\,|x(T)|/\bar{R}}\Bigg[|HA_2|\Bigg(|A_1|+|A_2|+(|B_1|+|B_2|+|B_3|)\\
&\times\left[\frac{e^{\gamma\tau}}{1+(e^{\gamma\tau}+1)\,|x(T)|/\bar{R}}+1\right]\sqrt{\varphi(H)}\,|x(T)|\Bigg)\tau\\
&+|H|\Bigg(|B_2|+\left[\frac{e^{\gamma\tau}}{1+(e^{\gamma\tau}+1)\,|x(T)|/\bar{R}}+1\right]\\
&\times|B_3|\sqrt{\varphi(H)}\Bigg)|x(T)|\Bigg]\Bigg\}|x(T)|^2.
\end{aligned}
$$

Finally, let $|x(T)|/\bar{R} \le \zeta$, $0 < \zeta < 1$. Then, we have

$$\begin{aligned}
\dot{v}\left(x(T),T\right) \le -e^{\gamma T}\left(1-\frac{|x(T)|}{\bar{R}}\right)^{-2} &\left\{-\gamma\lambda_{\max}(H)+\lambda_{\min}(C)\left(1-\frac{\tau}{\tau_0}\right)\right. \\
&-\frac{2z(\gamma)\sqrt{\varphi(H)}}{1+\zeta z(\gamma)}\left[|HA_2|\left(|A_1|+|A_2|+(|B_1|+|B_2|+|B_3|)\right.\right. \\
&\left.\times\sqrt{\varphi(H)}\,R\zeta\frac{2+(\zeta+1)z(\gamma)}{1+\zeta z(\gamma)}\right) \\
&\left.\left.+|H|\left(|B_2|+\frac{2+(\zeta+1)z(\gamma)}{1+\zeta z(\gamma)}|B_3|\sqrt{\varphi(H)}\right)\bar{R}\zeta\right]\right\}|x(T)|^2, \\
z(\gamma) = e^{\gamma\tau}-1. &
\end{aligned}$$

We shall find $\gamma > 0$ so that $\Psi(\gamma) > \gamma\lambda_{\max}(H)$, where $\Psi(\gamma)$ is defined in (13). In fact, if this inequality holds then the total derivative of the Lyapunov's function will be negative definite and $(x(t),t) \in v_t^\alpha$ at $t > 0$ as long as $(x(0),0) \in v_t^\alpha$. For this, we note that the function $\Psi(\gamma)$ is concave for $\gamma > 0$. Thus, from Lemma 3, for $\tau < \tau_0$, $0 \le \gamma \le \gamma_0$, and $|x(T)| < \bar{R}/\sqrt{\varphi(H)}$, the total derivative of the Lyapunov function is negative definite. Hence, the vector field of system (1) will be directed inside the domain v_t^α and $(x(t),t) \in v_t^\alpha$ for all $t > 0$. But, then the inequality (6) in Lemma 2 holds. Now we substitute α and ξ defined in (16), to obtain the solution convergence estimate (15).

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